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# Undergraduate Students' Cognitive Obstacles in the Learning Power Series Concepts Using APOS

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***Abstract***

Power series are a powerful tool to study elementary functions that are widely used in computational sciences to obtain approximations of functions. However, students do not fully develop required mental structures due to the presence of cognitive obstacles. The purpose of this study was to develop students' mental constructions in learning power series expansion using the activities, whole class discussions and exercises instructional approach. Literature on the activities, class discussion and exercise instructional approach in power series is scarce. This framework advocates for the use activities, class discussion and exercises teaching cycles to develop undergraduate students' mental constructions in mathematics concepts. A qualitative case study research approach involving 101 first-year undergraduate students was adopted for this study. Students' understanding of power series expansion was gathered by using a task-sheet and semi-structured interviews. Data analysis consisted of content analysis of students' written responses to the exercises and identify categories. The results indicated that students partially developed the schema for expanding power series and solving related problems. Students had some content gaps in the pre-schemas, which in true became some cognitive obstacles in learning power series. Future studies may focus on mental constructions and obstacles in other aspects of series expansion.

***Keywords:*** *APOS; ACE Teaching Cycles; Power Series; Digital Technologies; Genetic Decomposition; Undergraduate Students; Cognitive Obstacles*

## *Introduction*

The study of infinite series causes great difficulties in undergraduate students (Martínez-Planell, Gonzalez, DiCristina &Acevedo, 2012). These difficulties persist in students power series. Undergraduate students studying a calculus course at a university in South African do power series in the first year of their study. Power series can be used to define new functions that cannot be expressed in any other way than as "infinite polynomials". The basis of power series is sequences, which are naturally occurring numbers that follow a definite order. For every positive integer $n$, a sequence is denoted by $\{a_n\}_{n=1}^{\infty}$ or $\{a_n\}_{n=1}^{k}$for an infinite or finite sequence respectively. Adding terms of a sequence yields a series and is denoted by $\sum_{n=1}^{\infty} a_n$ for an infinite or $\sum_{n=1}^{k} a_n$ a finite series. Power series are strictly an infinite summation of terms of function $f(x)$ in which each term is in ascending powers of the variable $x$, which takes the general form

$$f(x) = \sum_{n=0}^{\infty} a_n(x-c)^n = a_n + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \cdots$$

where $a_n$'s are the coefficients of the series, $x$ is a variable and $c$ is an arbitrary constant (Stewart, 2010). Power series are centred at the arbitrary constant $c$. If the value of $a_n$ remains constant and the value of $c$ is set to $0$, the power series becomes the geometric series. Thus,

$$\sum_{n=0}^{\infty} ax^n = a + ax + ax^2 + ax^3 + \cdots$$

where each successive term increases by a common ratio of $x$. On the other hand, determining the coefficients of the series $a_n$'s yields the Taylor series of the function $f(x)$ at $x = c$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)(x-c)^n}{n!} = f(c) + f'(a)(x-c) + \frac{f''(a)(x-c)^2}{2!} + \frac{f'''(a)(x-c)^3}{3!} + \cdots$$

Thus, power series are closely linked to Taylor series. Taylor series can be used to approximate functions about $x = c$ and this is possible if the power series does not stray too far from its centre. For a special case $c = 0$ the Taylor series becomes

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)x^n}{n!} = f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \cdots$$

The function centred at called $c = 0$ is the Maclaurin series. It is used to approximate any function $f(x)$ about $x = 0$, with the result being more accurate when more terms are included in the summation.

Power series is one of the main mathematics at university which has wide application, particularly in approximations which transition students from school mathematics to university mathematics. In high school, South African students learn sequences and series, including summing infinite geometric series. The instruction of power series is deemed to be impeccable but the conceptualisation is varied. The goal of this study was to teach power series using the teaching cycles of Activities-Class discussion-Exercises (ACE) on the digital technologies platform. The teaching cycles were digital because in 2022, all teaching and learning at the institution was fully online due to effects of the Covid-19. The study sought to determine students' mental conceptions of power series expansions and classify them in order to understand the success of the instructional design (Tokgoz, 2018). In order to achieve this purpose, I formulated the following research questions: To what extent do the ACE teaching cycles facilitate learning of power series? What are the students' conceptions of power series expansion?

Many studies have been conducted to find the conceptualisations that students have with specific mathematics concepts using the Action-Process- Object-Schema (APOS) theory. However, few studies have been conducted on the way students coordinate knowledge to solve power series problems in the ACE teaching environment. This study contributes to that. The limitations of this study was that it was conducted at a particular time and a single case was considered. Hence, generalisations cannot be made directly to other settings; however the findings can be used to inform future instructional decisions (Kazunga & Bansilal, 2020).

**Literature and Theoretical Framework**

(In) finite series in calculus has been a subject of APOS theory studies. Tatira (2021) investigated binomial series expansion among year-two calculus undergraduate students and discovered that students can execute step-by-step in expanding binomial series. However, they grappled with the concepts calling for higher-level mental constructions and application of the binomial series expansion. By focusing on power series, this study goes beyond binomial series expansion. In addition, power series beyond expansion by creating new functions which are which are representations of the original function. Tokgoz (2018) used the APOS theory to investigate engineering students' knowledge of Taylor series expansion. Taylor series expansion is wholly part of power series expansion which has applications to physics and engineering. The findings revealed that students had well-established knowledge of approximations and infinity. However, they had poor knowledge of the meaning of centre $c$ concept that takes place in the Taylor series expansion of functions.

Power series are all expressible as infinite series, and expressing terminating functions or polynomials as a summation to infinity is a skill students are expected to grasp. Hence studies of students' understanding of the infinite series become prominent. In this regard, Martínez-Planell, Gonzalez, DiCristina and Acevedo (2012) conducted a study on graduate students' mental constructions of infinite series. Results from that study revealed that students have difficulties in constructing an understanding of series and thus tend to have difficulties in situations that require such interpretation. In most cases, all the studies considered in the literature review sought to understand students' mental reasoning in specific mathematics concepts as an initial step. Thereafter, these observations might help to develop a successful teaching methodology after strengths and weaknesses of the students have been investigated.

To analyse data for this study, the APOS theoretical framework by Dubinsky and Mcdonald (2001) was used. The APOS theory was chosen because it focuses on students' construction of knowledge and how instructors can use this information to advocate pedagogical actions to propel the learning process forward (Salgado & Trigueros, 2015). APOS represents the hierarchical cognitive development of students' mathematical knowledge as they learn a mathematical concept, which is first conceived as explicit step-by-step memorised facts with each step providing a cue to the next. Individual students interiorises actions into a process conception when they can repeat, predict and reflect upon each action without having to explicitly perform it. Encapsulation occurs when students conceive actions and processes as a totality and realise that transformations can act on the totality in new situations. Finally, a schema is a collection of actions, processes, objects and other schemas that are organised into a coherent framework. The framework enables students to decide on the appropriate mental processes to be used in dealing with a mathematical situation (Voskoglou, 2015).

According to Arnon et al. (2014), research and/or curriculum development based on the APOS theory is composed of the theoretical analysis, design and implementation of instruction, and collection and analysis of data. The diagram in Figure 1 illustrates the relation between the three components.

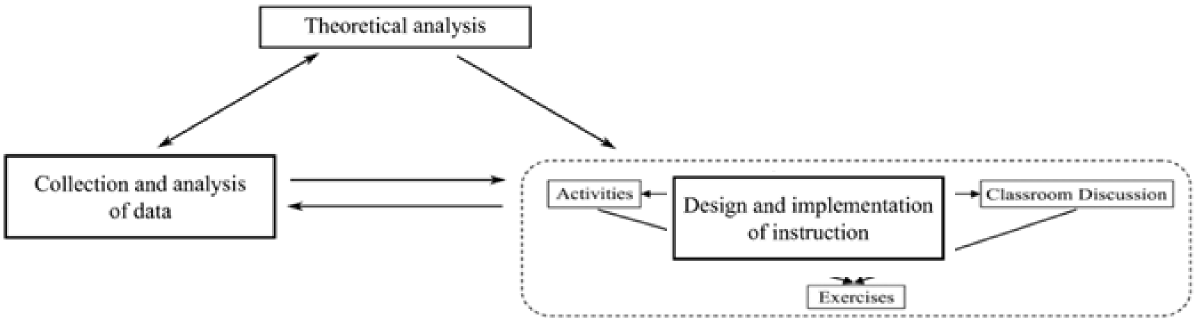


Figure1. APOS research and ACE teaching (modified from Asiala et al., 1996)

The APOS theory is used in research as an evaluative or instructional development tools or both (Brijlall & Ndlazi, 2019). In this study, I used it as both. According to Figure 1, research starts with a theoretical analysis, which is a description of the mental constructions that students might in making sense of a mathematical concept. The researcher has this responsibility to craft the hypothetical model. The model is called the genetic decomposition (GD), which is a function of the researchers' knowledge of mathematics, teaching experience and review of literature (Martinez-Planell & Delgado, 2016; Oktaç, Trigueros & Romo, 2019). The preliminary GD for this study is given in the next section.

## Preliminary GD

To build a schema for power series expansion, students are expected to possess the pre-schema of geometric series, differentiation, integration, laws of exponents and convergence of infinite series. Even though geometric series are not polynomials, they represent the basic type of power series where the coefficient is constant and the ratio between successive terms in the series is common. Everything about geometric series holds for power series by letting $|r| = |x|$. At action level, students are expected to expand standard functions and polynomials as infinite series using the Taylor's formula, without thinking beyond the functions being expanded. Students are also expected to determine power series representations of standard functions explicitly. Procedural knowledge should act as the foundation of all levels of mathematics thinking in mathematics (Kazunga & Bansilal, 2017).

As the understanding of these procedures deepen, students start to discern properties and relationships between concepts embedded in the procedures. When students reflect and repeat functions representations of power series, they can standardise given functions and find their power series representation without performing the memorised step-by-step operations. At this level, a students can work in reverse by finding the function given a power series representation. Students can also predict whether it is possible to expand any given function and expand generalised functions. In object-level conception students can see the effect of power series representation of any function as a totality. They can explain why it may be possible or not possible to expand given functions. Students are able to perform actions and processes together with other transformations on power series representations of given functions. For example, differentiation or integration of a power series expansion. Ideally, a GD depends on the researcher who designs it, hence is not constant; more than one GD of the same concept may exist (Borji & Martinez-Planell, 2020). The GD is cyclical so that it can be refined, tested and validated to make the proposed GD compatible with the research results obtained (Trigueros & Oktaç, 2019).

## APOS Theory and Implementation of Instruction

The second stage of APOS theory is designing and implementing proposed instruction as instituted by the GD. The GD directs the design and implementation of instruction through carefully selected activities and exercises. These are meant to develop the mental constructions called for by the GD. The ACE teaching cycles seek to help the students to construct expected actions, interiorise the actions into processes and encapsulate processes into objects. Equally, two or more objects can be coordinated to form new objects. The modified diagram in Figure 1 illustrates the ACE teaching cycles occurring at the node of Design and Implementation of instruction and directly informed by the GD. The ACE teaching is cyclical because the researcher continuously refines the instructional activities until the concept is understood and completed.

Finally, completion of instruction using the ACE cycles culminates into the collection and analysis of data, according to Figure 1. Collection and analysis of data purposes to ascertain the degree to which students attain the stated mental constructions. If students have not attained the requisite mental constructions, then implementation of instruction is reconsidered and revised. Moreover, the analysis of data checks mental constructions and cognitive obstacles that students encounter to learn a mathematical content. If there are misconceptions and cognitive obstacles, the GD is revised, hence starting another APOS theory cycle. The APOS cycles are repeated until students have successfully learnt the mathematical concepts. The cycle ceases when the theoretical analysis and the empirical evidence from instruction coalesce. The next section alludes the actual implementation of instruction.

## Methodological Foundations

This study used the interpretive research paradigm, which seeks to understand the ever-changing construction of reality in how students construct knowledge when learning a concept. The interpretive paradigm construe that individuals construct their own view of the world based on their perceptions and experiences. Researchers tend to “rely more upon the participants’ views of the situation being studied and recognises the impact on the research of their own background and experiences” (Creswell, 2003: 8). The interpretivist researcher relies on the use of qualitative methodologies and data analyses (Krauss, 2005). To gain an in-depth analysis of undergraduate students’ understanding of power series representations, a case study design was used. The identified unit of study was the first-year students at a university in South Africa. The participants were conventional students registered for a calculus course. The course had 101 students who all agreed to take part in the study.

The ACE teaching cycles were implemented in power series instruction under the guidance of the preliminary GD. The study was conducted in 2022 when teaching and learning was fully online and remote, hence digital technologies dominated teaching, learning and assessment in the course. The researcher created consecutive video recordings which gave concise explanation and activities on the different aspects of power series representations and expansions. These were uploaded Moodle for easy access by students prior to class sessions. Students went through the explanations and did the activities either individually or in groups before class times. Each recording was immediately followed by the whole discussion on the aspect under consideration on that day following the formal timetable. Classes were conducted virtually on the Microsoft Teams platform. The problems done by students during the activities stage of ACE were discussed at each turn of the cycle in the whole group discussion (Trigueros & Oktaç, 2019). At the end of the each class discussion, individual exercises were issued. The thrust of APOS theory is on doing activities as a way of constructing mathematical knowledge. APOS is premised on the constructivist learning approach whereby students construct new knowledge and understanding from their engagements with content material in form of activities (Afghani, Suryadi & Dahlan, 2017).

The administration of activities, class discussion and exercises were repeated until the concept of expanding power series was completed. The first cycle targeted base line understanding of power series expansion. The second ACE cycle was to facilitate the interiorisation of this action-conception to a process (Voskoglou, 2015). Similarly, the activities and the discussion of the third cycle was meant to facilitate the encapsulation of the concept of power series expansion. The last cycle was to help students to determine the conditions necessary to expand given functions including the interval of convergence.

Only one cycle of the APOS theory was done due to time constraints as the normal curriculum concerns were to be followed and completed in specific times. The series of activities, class discussions and exercises were designed to develop the mental constructions as informed by the GD and this became the first step in the ACE teaching cycle (see Figure 1). The task was administered under time-control as a Moodle assignment. The researcher graded the task using in-line grading on Moodle and the annotated scripts were availed to students as soon as the grading was complete.

Based on Figure 1, the implementation of instruction gives way to data collection. The data determine "how learning has taken place─evaluating the instructional approach─as well as testing the genetic decomposition" (Arnon et al., 2014, p193). Data were collected through written responses to the task of the 101 students. Student' written responses were analysed carefully in an attempt to determine participants' understanding of the concept of power series. The students were assigned pseudonyms *B1*, *B2*, *B3* and so on until *B101* for confidentiality, where the ordering did not have any significance. After the scripts were marked, nine students volunteered to be interviewed telephonically and follow-up questions were asked depending on their initial responses to the tasksheet. Semi-structured interviews were audio-taped and transcribed. The interviews captured deeper insights into participants' experiences and understandings gained through the activities, class discussion and exercises.

The task-sheet consisted of three problems focusing on various aspects of power series representations and expansions, which were planned based on the preliminary GD. These are:

1. *Find the Maclaurin series expansion for* $(1+x)^m$ *up to the term in* $x^3$.
2. *Find a power series representation for the function* $\frac{8}{2x-15}$ *using summation sign.*
3. *Find the power series representation of* $\frac{1}{(1+x)^2}$.

Question 1 required process understanding of expanding power series of a generalised function. Question 2 was at process level where students had to standardise the function first physically or mentally before they apply the action understanding to obtain the simplified representation. Question 3 appealed to the object mental construction whereby students have to identify the need to apply derivatives to the function and then perform actions and processes to simplify the representation using appropriate the alternating series formula.

Attempts were made to present data in a way that communicates as much information as possible and analyse it so that meaning can be attached to the data (Bertram & Christiansen, 2014; Corbin & Strauss, 2008). Data analysis was based on identifying themes, similarities and patterns emerging from data based on APOS mental constructions. Cognitive obstacles arising from students' written responses were also identified where appropriate.

## *Findings*

The first step in data analysis involved categorising all written responses as no response, and incorrect, partially correct and fully correct responses. Secondarily, an in-depth content analysis was done in addition to the double dichotomous coding above. The interview transcripts were concurrently analysed so that the results could be complemented. Detailed analysis of data is presented in the next section. The results of this study are reported in terms of power series expansion (question 1) and representation (questions 2 and 3). The frequencies of students' responses codes for all the questions are shown in Table 1.

Table1. Frequencies of students' responses in the task-sheet questions

| Categories/Item | 1 | 2 | 3 | Totals |
|---|---|---|---|---|
| No response | 53 | 10 | 6 | 69 |

| | | | | |
|---|---|---|---|---|
| Incorrect | 31 | 54 | 75 | 160 |
| Partially correct | 5 | 22 | 19 | 46 |
| Fully correct | 12 | 15 | 1 | 28 |
| Totals | 101 | 101 | 101 | 303 |

The scores from this task did not contribute to the coursework hence students were at liberty to respond or not to. Consequently, there were sizeable blank spaces left in the scripts of some students, which I conclude as evidence of lack of how to attack a question. It could also mean that they ran short of time to answer all questions. Upon further inquiry in the interviews, B10 and B20, who both skipped questions 1 and 2, said:

*B10: Yes, we had covered similar questions in class but I was not sure where to start for both questions. You were supposed to give hints us on both questions, Sir.*
*B20: I lacked confidence and decided not to respond Sir. But the questions were fair.*

Moreover, more than half of the students attempted the questions but did not meet the minimum requirements to get even part marks. These were actually 160 out of 303. Generally, the frequency for the no and incorrect responses was much more than the frequency for those who got partial and fully correct responses. This was evidence of challenges to learning power series, which I classify as cognitive obstacles. The content analysis of written responses identify the nature of these. The fully correct responses registered only a frequency of 28 out of 303, which is nine percent of all the responses. Nevertheless, the APOS theory goes beyond simple success and failure in answering questions; it checks for the development of the mental constructions called for by the GD and their shortcomings. The content analysis elaborates this in the next section.

*Question 1*

This question was intended to invoke students' process skills in expanding the general function $(1+x)^m$. After reflecting on the generalised form of the question in their minds or on paper, students could then apply the step-by-step manipulation of the procedure to expand using the known Maclaurin's formula. Firstly, five students did not differentiate the function at all, which is the key to the use of Taylor/Maclaurin's formula. Without repeated differentiation of the function, constructing the Maclaurin series expansion is not possible. Figure 2 shows the pre-action of series expansion which lacks the step by step procedures and operations.

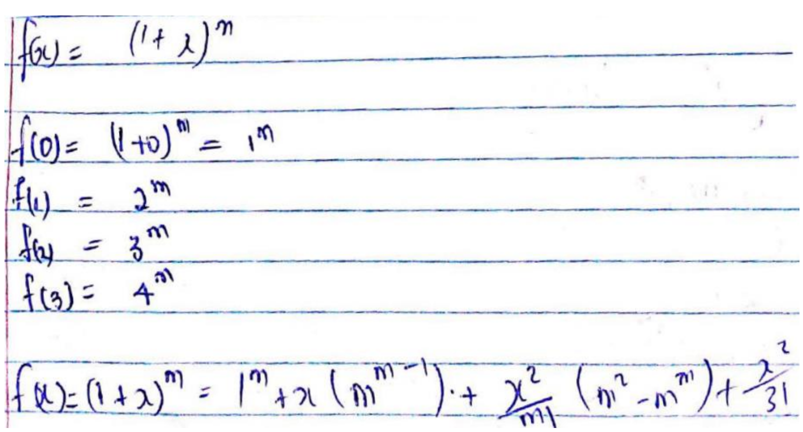

Figure1. Evidence of lack of action conception of power series by B97

On the other hand, five students attempted to differentiate but without success, as shown in Figure 2. For these, the pre-schema of differentiations was not encapsulated. The existence of a general

International Journal of Social Science Research and Review
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December, 2022

function should not inhibit students from executing correct differentiation. However, by considering Figure 2 and the solutions of four other students, they could not manipulate derivatives involving the $m$ in the exponent as it should be.

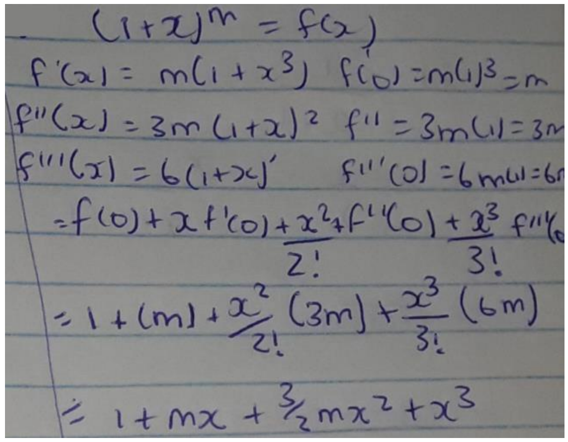

Figure2. Incorrect expansion as a result of failed differentiation by B94

In the interviews concerning the exponent *m* in the exponent, B50 indicated that he treated it as a variable. However, this should not be the case to a student who has interiorised power series expansion; the question specified that the expansion should be up to the term in $x^3$. Only $x$ was the variable in this single-variable function. The presence of a $m$ in the exponent led to ten students failing to accurately substitute $x = 0$ in $f(x)$ or in the derivatives of $f(x)$. This difficulty was evidence of lack of action understanding of power series expansion. The pre-schema for exponents was not fully developed. For example, some students did not show realisation that $1$ raised to any power remains $1$. In Figure 3, the error was consistent in all of $f(0), f'(0)$, and so on, indicating it was systematic.

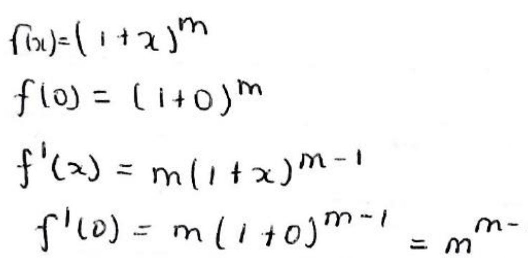


Figure3. Lack of schema for exponents by B46

Moreover, B14 evaluated correct derivatives but encountered obstacles in evaluating $f(0), f'(0)$, and so on. Substitution in formula is an action-level conception according to the GD. However, 19 participants performed flawed substitutions in the Maclaurin's formula. Of these, 13 substituted in the wrong formula, as they could not recall the correct one. In the interview, B23 said that he knew the formula to use but was confused by the $m$ in the exponent. Having successfully computed the derivatives at $x = 0$, some students could not substitute into formula. This represented lack of action conception of power series expansion.

Students who made the diverse kinds of errors mentioned above in question 1 were 36. Also from Table 1, it was shown that 53 students did not attempt the question, which I conclude to denote possible lack of action conception of power series expansion. Most concepts are first conceived at the action level (Voskoglou, 2015; Mutambara & Tsakeni, 2022), hence failing to answer a question might mean absence of such. Finally, twelve students managed to respond correctly to this question. These managed to expand a general function, which required process level of understanding. Figure 4 illustrates a correct response to question.

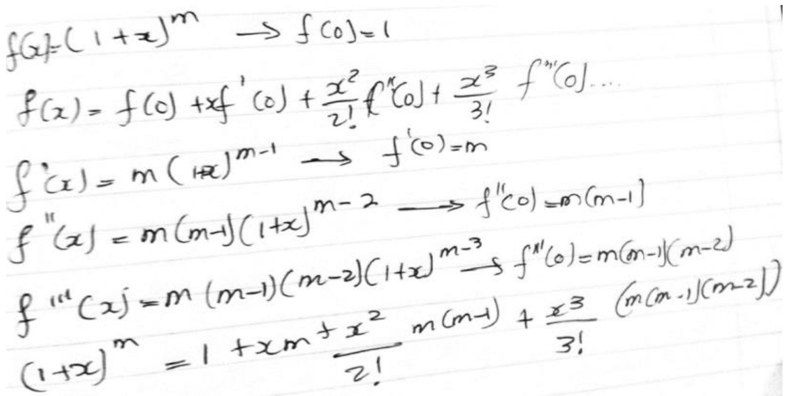

Figure4. A perfectly correct answer by B67

The student successfully evaluated the first three derivatives since the expansion was up to and including the term in $x^3$. Thereafter, she computed the derivatives at $x = 0$ according to the Maclaurin formula. This demonstrated the Process skills in power series expansion. The majority of the students also managed to identify that the expansion was centred at 0. This was implicitly given, hence it represents growth of process mental construction.

*Question 2*

This question required object conception of power series in that students had to standardise the function first before applying power series expansion to the function by making a comparison to the standard power series expansion without physically doing the steps. Fifteen students managed to attain object mental constructions as shown in Figure 5.

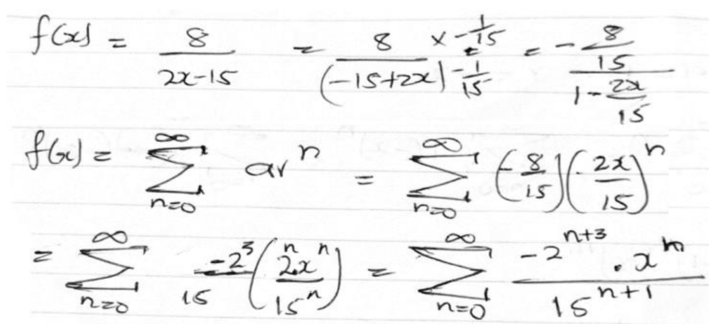

Figure5. Encapsulation of power series expansion by B38

More than half of the students did not develop mental constructions to at most action level. They operated at the pre-action level. This problem was supposed to be standardised first in order to compare to the standard $\frac{1}{1-x} = a + ax + ax^2 + ax^3 + \cdots = \sum_{n=0}^{\infty} ax^n$. As many as 32 students did not standardise the function at all. These 32 also included those who performed wrong standardisation processes. By so doing, they could not get the expected power series representation of the function. The act of standardising is not externally motivated, but entail anticipating and working in reverse, sometimes mentally. For instance, the students who managed to standardise had no problems in determining the power series representation.

Some 22 students who got partially-correct responses and some who got wrong responses according to Table 1 encountered problems with the standardisation process. Failure to get the correct standardised form of the function led to a miscarriage in getting the power series representation. For example, 31 students started the standardisation in the right way but had challenges in simplifying the expression. Of these, eight did not factor consider the coefficient $-\frac{8}{15}$ at the last step. Figure 7 illustrates this error. They completely failed to merge $-\frac{8}{15}$ and $\left(\frac{2x}{15}\right)^n$ to obtain the required $\frac{-2^{n+3}x^n}{15^{n+1}}$.

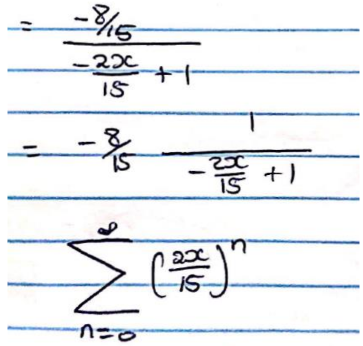

Figure7. Error of omitting $-\frac{8}{15}$ in the final answer by B23

Upon inquiring how such a thing could happen, B76 responded that he forgot the term entirely, whilst B23 said "*I wanted only the power series representation for the function and I got it as shown after the summation sign*". Unlike B23, four students neglected the 8 in the numerator in the process of standardisation, leading to a partial answer. In fact, executing algebraic processes and simplifying exponential terms was a cognitive challenge to 32 students. The pre-schemas in algebra and exponents were not fully developed. In Figure 8, B99 committed two errors of leaving out 8 and of removing the brackets in the numerator.

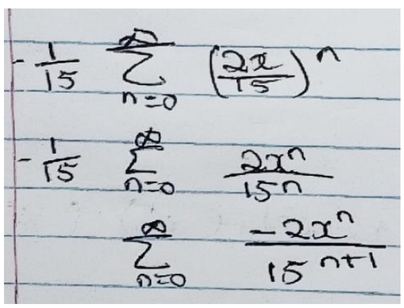

Figure8. Incorrect simplification and a missing factor in the expression by B99

B95 also made the same mistake in the numerator and ended up getting -16 after multiplying by -8. Besides that, some students did not factor out the negative sign in the denominator to match the standard expression for power series representations. The response by B98 in Figure 9 illustrates this error.

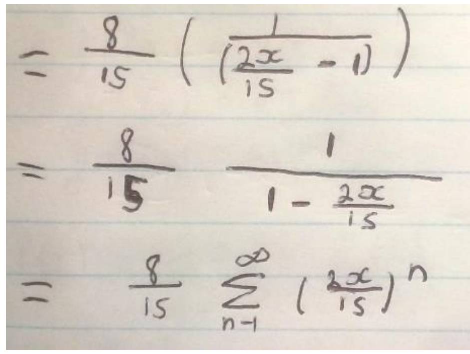

Figure9. An error of losing a negative sign by B98

Successful action-level conception of power series should see students do precise step-by-step algebraic simplification of the expression. Representing functions as power series relies on these steps to get the correct representation.

A number of some students did not make attempts to standardise the function at all and some did not use the power series approach to solving this problem. Also, B97 did not use the concept of power series, when yet the question was specific as shown in Figure 10.

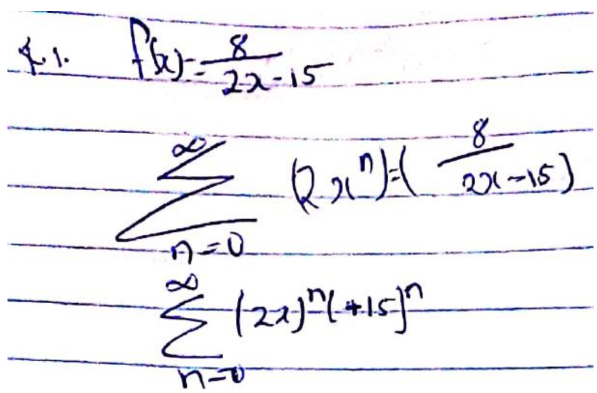

Figure10. Evidence of not using of power series by B97

Also, more than twenty students paid complete disregard to the condition of convergence powers series in their efforts to standardise the function. Infinite power series expansions are convergent if and only if $|x| < 1$. Indeed students like B27 even computed the interval of convergence, but could not put it into context (shown in Figure 11). They expanded the function in powers of $(x - 5)$ instead of $x$. Furthermore, B27 showed the procedure to get the interval of convergence as $|x| < \frac{5}{2}$.

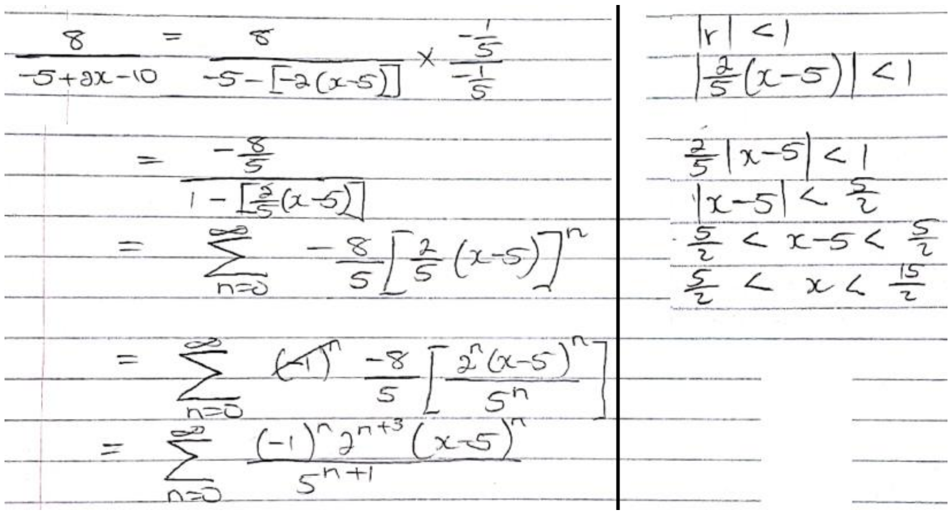

Figure11. Power series expansion that disregards the interval of convergence by B27

In Figure 11 B27 has perfect knowledge action conception to find the interval of convergence but given as greater than 1. It illustrates lack of sufficient object conception of power series expansion, which was not understood in its entirety. The approach to standardisation is correct but leads to a diverging series since $|x| > 1$. Upon further inquiry, B27 said he did find it normal to standardise the denominator to $1 - \left[\frac{2}{5}(x-5)\right]^n$ and paid no attention to the meaning of interval of convergence. B27 did not encapsulate the procedure of power series expansion

*Question 3*

Question 3 in the task required students' object conception upon which the processes of differentiation and the alternating series formula could then be determined. These may be done physically or mentally, depending on students' mental constructions. Only B46 developed required mental structures to perform these processes precisely, as shown in Figure 12.

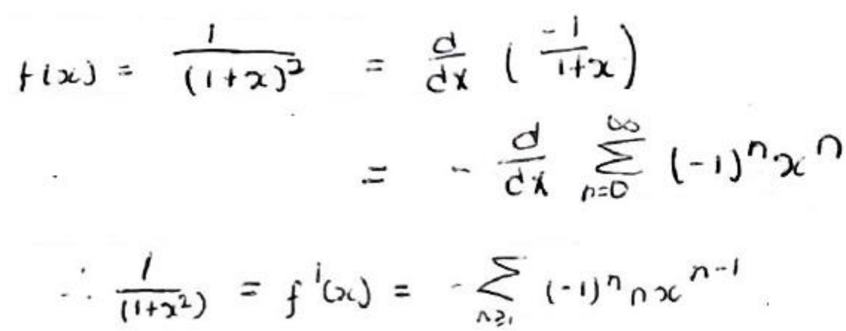

Figure12. The correct processes of differentiation and alternating series by B46

B46 interiorised differentiation of reciprocal functions and encapsulated the alternate series formula. Six students skipped this question while 75 attempted it but did not obtain the correct solution as shown in Figure 12. The greatest mistake was failure to differentiate the function in the first place, which was done by 22 students. A further 12 students changed the original question to a simpler one. This way, they ended up with the summation of $(-x)^{2n}$ and the solution no longer required differentiation as was in the original question.

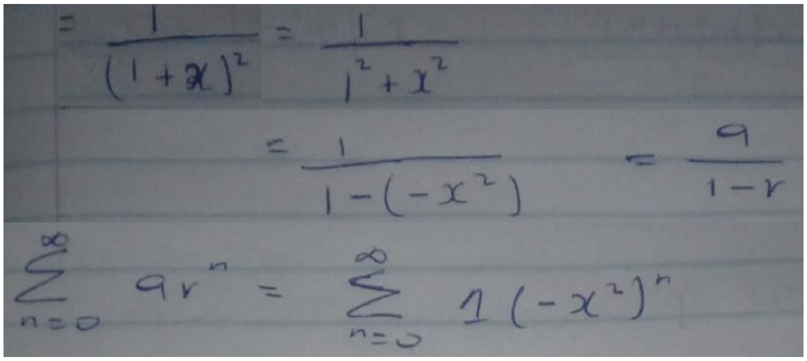

Figure13. A flawed simplification which led to a simpler question

Identifying the function as an anti-derivative was to key to the solution of the problem. Without actually doing it, students were required to identify that $\frac{1}{(1+x)^2}$ is the derivative of $-\frac{1}{1-(-x)}$. However, the actions and processes needed to get the power series representation were not fully developed. In this regard, 22 students lacked the process conceptualisation as evidenced by missing the negative sign to the derivative of $\frac{1}{1+x}$. This is shown in Figure 14.

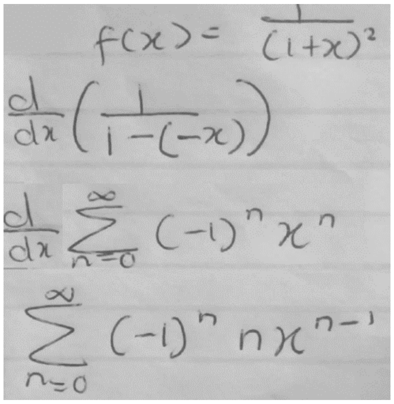

Figure14. Missing negative sign in the differentiation process by B81

Process-level conception is needed to find the representation of $\frac{1}{1-(-x)}$ using the alternating series formula. Nine students did not have $(-1)^n$ as expected in this instance. After realising a negative sign is needed in the representation of $\frac{1}{1-(-x)}$, 14 students expressed it as $\frac{d}{dx}(\sum(-x)^n)$ as shown Figure 15.

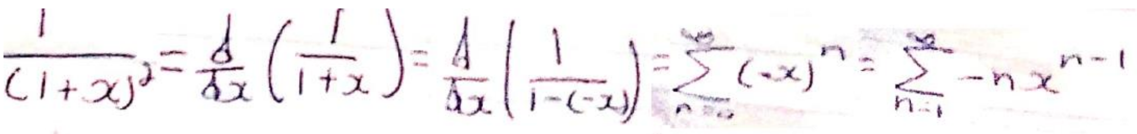

Figure15. Lack of understanding of the alternating series formula by B86

The interview with B24 confirms the incomplete understanding of the alternating series formula.

*Researcher:Have you ever heard of the alternating series formula?*

*B24: Yes.*

*Researcher:Are we not supposed to get $(-1)^n$ is a series is alternating?*

*B24: There is no need because $\sum(-x)^n$ and $\sum(-1)^n x^n$ are the same.*

*Researcher: Are you sure*

*B24: Yes. Both yields the same alternating series Sir.*

According to B24, the alternating series can be represented as $\sum(-x)^n$, which differ from the established formula according to calculus. This error in turn affected the accuracy of the derivative of the summation. Students who had challenges with alternating series such as B24 perceived power series representation as an action and did not have the process conception of the representation.

## *Discussion*

To develop the schema for power series expansion, students need to attain the object mental construction. Students first conceive a concept at the action level (Voskoglou, 2015) by building upon pre-schemas and other prior knowledge. Attaining the objection mental is the principal theme in mathematics instruction. With the aid of the GD, I created teaching activities that opened opportunities for students to reflect upon their actions and develop process skills (Oktaç, Trigueros & Romo, 2019). To help achieve that, two items in the task-sheet were pitched at the object level and one at process. Task design in mathematics is intended to students' consolidate knowledge, engage students in activities and reveal students' conceptions and obstacles (Trigueros & Oktaç, 2019). Task design is at the heart of APOS-ACE instruction to develop students' mental constructions and reveal the extent of the transformations acting on these constructions. However, the journey from action to object mental constructions is fraught with many cognitive obstacles. The discussion of students' mental constructions and cognitive obstacles are elaborated in detail below.

The findings revealed that when students fail to attain the process and object mental constructions, the action transformation that were supposed to follow them failed to materialise too. In items 1 and 3, if students failed to differentiate the functions, the ensuing physical step-by-step expansion and representation could not be achieved respectively. Without differentiating, the Maclaurin theorem is inapplicable. In item 1, students lacked the action-level conception to differentiate a function. The constant $m$ was mistaken for a variable. That was a huge obstacle to many students. Had it been a normal specific function, the majority of students would have managed since only action conception would be called for. Oktaç, Trigueros and Romo (2019) also discover that students were unable to solve using generalised approach in their responses. In item 2, students who could not encapsulate power series representations by reducing the function to the expected standard form in the light of conditions of convergence of infinite series. Students are expected to physically or mentally reduce the given function to standard form of power series expansion by anticipating the outcome and working backwards. However, for some students, this was a mental construction they failed to attain.

For item 3, the only way to change the function to standard was to identify that the given function was an anti-derivative of a standard function $\frac{1}{1-(-x)}$. This required understanding power series expansion as a totality whereby further transformation were supposed to be done. For example, students had to introduce the negative sign as required after differentiating $\frac{1}{(1+x)^2}$. It was a cognitive obstacle for some students as they went ahead and wrote the represented the power series without the negative sign. Getting the correct differential for the given function was an action transformation which some students could not do. In this case, when the higher-order mental construction were not achieved, then the low-order skills too were not achievable. This is true for problems which require a total understanding of a concept. In most studies on APOS theory, students find it easy to demonstrate their understanding of a concept at action level if the problem is explicit and only require physical step-by-step solution (Borji, Alamolhodaei & Radmehr, 2018; Tatira, 2021; Kazunga & Bansilal, 2020; Mutambara & Tsakeni, 2022). Moreover, the nature of some functions require the use of appropriate alternating series formula. These start with $(-1)^n$ but it was not the case with some students. Their understanding of alternating series was not fully developed in this regard.

Schema development for power series expansion hinged on successful understanding of the pre-schemas. It was explained earlier that a schema is a coherent collection of action, process and object mental collections and other schema. If pre-schemas are absent, schema development for the current concept may not fully develop. In power series expansion, the pre-schemas were knowledge of differentiation, exponents, geometric series and algebraic simplification. Many students had serious challenges in power series due to ill-developed pre-schemas.

Students in this study faced some cognitive obstacles in learning the concept of power series. The total percentage for the incorrect responses was 53 relative to the nine percent for students who got all the questions correct. This testifies that students had some challenges in power series expansion. The content analysis of the written responses revealed some content gaps in students' understanding of power series. The frequency of unattempted question was 23 percent, which indicates pre-action mental constructions. Students lacked the confidence to put into action their knowledge of power series; this is common when students are not sure of how to attack a question. Kazunga and Bansilal (2017) classified un-responded questions as a result of students being "blank" at that moment and had forgotten how to work out the problem. However, some of these obstacles were mitigated by the ACE teaching approach used in this study. By prioritising learning through doing and discussing problems section after section until instruction of a concept is accomplished, ACE teaching cycles really facilitated learning of power series. Use of digital technologies to administer the teaching cycles by heightened students' interest in engaging with content.

## *Conclusion and Recommendations*

This study considered cognitive obstacles in learning power series expansion because the task was poorly performed. The obstacles obviously mitigated students' schema development in power series. Students did not demonstrate sufficient interiorisation and encapsulation mental mechanisms of the concept of power series. Without a clear mental and total understanding of power series expansion, students could not perform further transformation on powers series expansion, even at the action mental structures. The greatest challenge in doing power series representation is to be able to express the given function in the standard for series expansion to infinity. This activity called for both process and object level conception. Hereby students are expected to predict, anticipate and manipulate functions in totality by incorporating conditions of convergence.

The majority of students fail to attain the object skills or took long to develop them due to presence of obstacles to cognitive development of a concept (Voskoglou, 2015). Oftentimes students display conceptual errors through failure to grasp concepts and find relationships in a problem (Chikwanha, Mudavanhu & Chagwiza, 2022). In case where the action skills followed directly the process or object, they were un-attainable too. In some cases, application and simplification of the power series representation using the alternating series formula was not achieved. Students were supposed to recall the formula and perform some steps to simplify the expression, but was not the case (Chikwanha, Mudavanhu & Chagwiza, 2022). Memorising procedures and imitating solution steps might lead students to succeed in the concept, but if the basic structures are not constructed, these strategies fail at some point (Oktaç, Trigueros & Romo, 2019). Tall (2004) posits that individual may become a way of life which may give success but only in routine contexts. But long-term Piagetian vision where operations become objects of thoughts require transformation of knowledge into thinkable mental entities.

The preliminary theoretical analysis of power series instruction still held; no modifications were not necessary thereto. The mental constructions compares well with the postulates in the GD (Oktaç, Trigueros & Romo, 2019). The researcher who is also the instructor identified the mental structures that might be needed in learning power series expansion sufficiently well. He also planned and enacted the activities that help students make the proposed mental constructions come true (Arnon et al., 2014).

APOS theory has been widely applied in many studies in mathematics but the ACE teaching cycles are less common (Borji, Alamolhodaei & Radmehr, 2018). The ACE instructional strategy focuses on how teaching can impact students' understanding of a concept. The ACE teaching cycles were successfully executed and the repetitions helped to clear some possible obstacles. Similarly, Borji and Voskoglou (2017) reveal that the implementation of APOS-ACE cycles benefits students much more than the traditional lectures did. The APOS ACE instructional approach based on the digital platform countered the low levels of students' engagement in learning mathematics (Syarifuddin & Atweh, 2022). Students became more engaged their learning and the aspects of anxiety and frustration decreased. Nevertheless, the activities, class discussion and exercises were more individualistic approach. Students at liberty to learn remotely at that time. And again the lecturer was a dominant figure in the class discussion as students were reluctant to un-mute and put forth their ideas. This is in contrast to the precepts of APOS didactical strategies such as using collaborative group work before and after concepts are formally introduced. Remote online teaching and learning by its nature does not promote cooperative learning and this study took place under those conditions. However, this was the best that could be done during full remote learning in the aftermath of Covid-19 pandemic.

This study helped to fill the gap of how students learn mathematics concepts in terms of the mental constructions and obstacles they are supposed to overcome to understand the concepts. Power series was chosen because doing representations of power series is not routine and procedural. Power series are a powerful tool to study elementary functions that are widely used in computational sciences to obtain approximations of functions (El-Ajou, Arqub, Zhour & Momani, 2013).

Course instructors should affirm in detail the prerequisite knowledge in developing action mental constructions in students as these are quite essential (Mutambara, Tendere & Chagwiza, 2020). The case is that all advanced concepts are based on some elementary concepts (Siyepu, 2015) thus students need assistance to understand the basic principles of power series in order to appreciate its application expansion and representation for instance. Thus, revisiting the pre-schemas on exponents, differentiation, geometric series and algebraic simplification is essential. Celik (2015) argues that course instructors should strive to provide appropriate pedagogical support to enable students to fully understand a mathematical concept. Instructors should be made aware of the significance of using carefully selected activities in the teaching and learning of mathematics (Samkova, 2012). This will help to address students' cognitive obstacles. The knowledge of these obstacles is important because it can provide guidance to the instructor in structuring the learning materials so that they can take these unexpected obstacles into account and help improve students' learning (Mutambara & Bansilal, 2022; Oktaç, Trigueros & Romo, 2019). The APOS-ACE teaching cycles in mathematics are useful in order to refine, detect and explain students' ways of thinking, their difficulties and connections among different constructions they can or cannot make (Trigueros & Oktaç, 2019). Future studies may focus on mental constructions and cognitive obstacles in other aspects of power series like tests of convergence and operations on power series.

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