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Role of Rectangular and Square Matrix in Graph Theory

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Abstract

Graph theory is one of the topics studied in modern mathematics and the concept of graph in mathematics was discussed from the past to the present. For example, graph relations and functions have many uses. The 18th century is the beginning of modern graph theory. From the 19th century onwards, graph theory was discussed for its application in various fields, which is a field of research at this time. Graph theory and matrix are two important and well-known topics in modern mathematics, so in this article the role of rectangular and square matrix in graph theory is discussed. The main purpose of preparing this article is to study rectangular and square matrix in graph theory. The research method in this article is library that was followed by using academic articles in libraries and the Internet. The findings obtained from this study show that the matrix adjacent to the graph is a square matrix and the occurrence matrix is a rectangular matrix.

Keywords: *Matrix; Graph; Adjacent Matrix and Occurrence Matrix*

Introduction

The word matrix was first used in 1850 by the English mathematician James Joseph Sylvester. For the first time, an English mathematician named Kylie Matrix has been introduced to mathematics. Many algebraic subjects such as linear algebra, algebra and vectors can be expressed with the help of matrices. Matrices are the basic tools of practical mathematical calculations today. The first article on graph theory was written by the famous Swiss mathematician Euler in 1736. Today, graph theory is used in various fields of study such as economics, psychology and biology. This article briefly explains the role of rectangular and square matrices in graph theory (Nazary, 2015b).

The role of rectangular and square matrices in graph theory has not been scientifically researched in Afghanistan's higher and semi-higher education institutions. Therefore, there was a need for a special research, so that we could ask which matrix is a rectangular graph and which is a square? Let's answer. The importance of this research is that matrix and graph are two important parts of mathematics. Therefore, one graph can be attributed to each graph, which shows the role of matrix in graph theory. For research in each section, it is necessary to refer to different sources and use the accurate works of scientists.

Therefore, in order to compile this article, it was researched based on complex library method (quantitative and qualitative). The graphs in question here are without multiple edges and without simple graphs, which is very precious for students and lecturers of mathematics. This article includes the basic concepts of graph theory, conclusions and references, which was compiled and edited by the researcher.

Definitions

Def. 1: Suppose $V \neq \varnothing$ is an infinite set and each elements of E is an ordered pair of $V \times V$ elements, the (V, E) ordered pair is called a graph and it is denoted with $\Gamma = (V, E)$ in which V is vertices set and E denotes the edges set of Γ.

Def. 2: Suppose Γ is a graph with V vertices set and E is its edges set, in this case:

1. If v_i and v_j are the two vertices of Γ and (v_i, v_j) is an edge of it, then v_i and v_j are called adjacent in which v_i is starting vertex and v_j is ending vertex.
2. The number of edges which pass through v_i vertex is called v_i, and denotes by $\deg(v_i)$. A vertex is singular if the degree of that integer be singular and a vertex is called pair if the degree of that integer be pair. In each graph, the number of vertices with singular degree is a pair integer. The number of Γ graph vertices is called Γ order and denotes by n and the number of Γ edges is called Γ volume and denotes by m (Nazary, 2015b).

Def. 3: In Γ graph, e edge is said to be directed, if its direction is specified from v_i vertex toward v_j vertex or vice versa. The directed edge denoted by $\rightarrow$ notation and it is called an arc. Otherwise the e edge is called undirected. There are two types of degree in directed graph: the out-degree vertex in a directed graph is equal to the number of edges which pass from that vertex. In the in-degree the vertex is equal to the number of edges which enter to that vertex.

Def. 4: The Γ graph which all its edges are directed, is called directed graph. Otherwise, Γ is undirected.

Def. 5: The Γ graph is called regular -r, if each vertex of it be equal to r.

Def. 6: n vertex and the regular -$(n-1)$ are called complete graph and it is denoted by K_n.

Def. 7: Suppose that Γ is a graph, v_i and v_j are its two vertices, a path with a length of l from v_i to v_j infinite vertices of row and Γ edges are as $v_i = u_0, e_1, u_1, e_2, u_2, e_3, \cdots, e_1, u_1 = v_j$. Thus for each $(1 \leq t \leq l)$, u_{t-1} and u_t vertices are adjacent. In this definition the repetition of edges and vertices are allowed.

Def. 8: Closed path in Γ graph is in a form of $v_i = u_0, e_1, u_1, e_2, u_2, e_3, \cdots, e_1, u_1 = v_j$. Thus $v_i = v_j$.

Def. 9: A trail in which neither vertex nor edge are repeated is called path. A n path vertex denotes by P_n. The number of path edges is said to be its length. Each vertex is a path of zero length. Also $u_0, u_1, \cdots, u_n$ path is called cycle, if $u_0 = u_n$. A n vertex cycle denotes by C_n. A cycle length is equal to the number of its edges. A cycle is singular or pair, if the number of edges be single, then the cycle is single otherwise, it is pair (Nazary, 2015b).

Def. 10: Suppose that Γ is a graph with set vertices of $V=\{v_1,v_2,\cdots,v_n\}$ (Haji, 2001). In this case:

1. The matrix $A=(a_{ij})$ is called the adjacency matrix of Γ which is a square matrix and its a_{ij} th element is:

$$a_{ij}=\begin{cases}1 & if\ v_i\ and\ v_j\ be\ adjacent\\ 0 & Otherwise\end{cases}$$

2. Suppose Γ is a graph with set vertices of $V=\{v_1,v_2,\cdots,v_n\}$ and $E=\{e_1,\cdots,e_m\}$ is its edges (Doosti, 2011).

The matrix $X=(x_{ij})$ is said to be occurrence matrix of Γ which is a rectangular matrix and its x_{ij} th element is:

$$x_{ij}=\begin{cases}1 & if\ v_i\ is\ the\ vertex\ of\ e_j\\ 0 & Otherwise\end{cases}$$

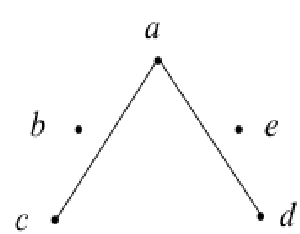
Here the sum of X i th row is equal to the degree of v_i vertex and the sum of each X column element s are equal to 2. In addition, the multiplication X i th row to the X i th row is equal to v_i vertex and the multiplication X i th column to the X i th column is equal to 2. If $i \neq j$, then it is written:

$$X\ i^{th}\ row\ X\ j^{th}\ row=\begin{cases}1 & if\ v_i\ and\ v_j\ be\ adjacent\\ 0 & Otherwise\end{cases}$$

$$X\ i^{th}\ column\ X\ j^{th}\ column=\begin{cases}1 & if\ e_i\ and\ e_j\ include\ joint\ vertices\\ 0 & Otherwise\end{cases}$$

We result from the above explanations that the adjacency matrix of Γ is rectangular and corresponding and its occurrence matrix is a rectangular matrix (Omit & Tajbaksha, 2007).

Example 1: Obtain the adjacency matrix of the following undirected graph.

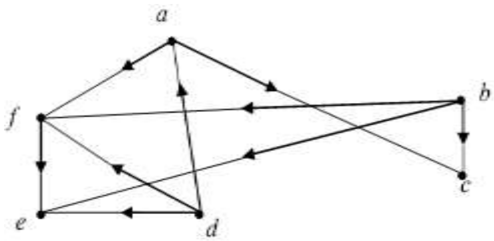


Solution: The above adjacency matrix graph is:

$$A=\begin{pmatrix}0&0&1&1&0\\0&0&0&0&0\\1&0&0&0&0\\1&0&0&0&0\\0&0&0&0&0\end{pmatrix}$$

The definition of adjacency matrix in directed and undirected graphs is similar.

Example 2: Obtain the adjacency matrix of the following directed graph.



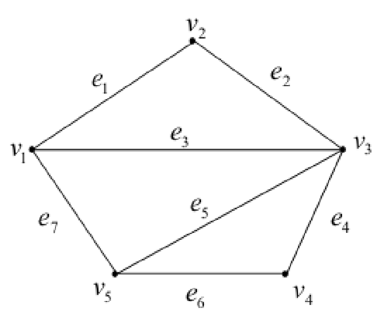
Solution: The above graph adjacency matrix is:

$$A = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 \end{pmatrix}$$

Adjacency Matrix Properties: Suppose that Γ is a graph with vertex n and edge m and A is its adjacency matrix, the basic properties of A are:

1. Each A element is zero or one.
2. The A main diagonal elements are zeroes.
3. The matrix A is square.
4. The matrix A is corresponding.

Example 3: Determine the occurrence matrix of the following undirected graph.



Solution: The above graph occurrence matrix is:

$$X = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

The Undirected Graph Occurrence Matrix Properties: Suppose that Γ is an undirected graph with vertex n and edge m and X is its occurrence matrix. The basic X properties are:

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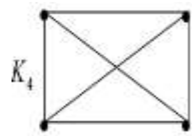
1. The X elements only consist of zero and one.
2. The sum of each X row elements is the degree of that row.
3. The sum of each X column elements is equal to 2.

Example 4: If A is the adjacency matrix of the complete graph K_4, indicate that the sum of matrix A^2 elements are equal to 36.

Solution: We consider the complete graph K_4, then we obtain the matrix of A.



$$A=\begin{pmatrix}0&1&1&1\\1&0&1&1\\1&1&0&1\\1&1&1&0\end{pmatrix}\Rightarrow A^2=\begin{pmatrix}3&2&2&2\\2&3&2&2\\2&2&3&2\\2&2&2&3\end{pmatrix}$$

The sum of matrix A^2 elements $=(3+3+3+3)+4(2+2+2)=4\cdot3+4\cdot6=36$

Example 5: If A be the adjacency matrix of the complete graph K_n, indicate that the sum of matrix A^2 elements is equal to $n(n-1)^2$ (Redmond, 2002).

Solution: The K_n adjacency matrix is:

$$A=\begin{pmatrix}0&1&1&\cdots&1\\1&0&1&\cdots&1\\1&1&\ddots&\cdots&1\\\vdots&\vdots&&0&\vdots\\1&1&1&\cdots&0\end{pmatrix}=A^2=\begin{pmatrix}n-1&n-2&n-2&\cdots&n-2\\n-2&n-1&n-2&\cdots&n-2\\n-2&n-2&\ddots&\cdots&n-2\\\vdots&\vdots&&n-1&\vdots\\n-2&n-2&n-2&\cdots&n-1\end{pmatrix}$$

In matrix A^2 the main diagonal elements is $n(n-1)$, the sum of each row without occurred element is on the main diagonal $(n-1)(n-2)$, and its entire elements row sum is $n(n-1)(n-2)$, in this case:

The sum of matrix A^2 elements $=n(n-1)+n(n-1)(n-2)=n(n-1)^2$

Def. 11: The determined polynomial adjacency matrix of the Γ graph was defined in the form of $\det(\lambda I-A)$ and denotes with $\chi(\Gamma,\lambda)$. Therefore its entire determined values which is the determined polynomial roots of Γ adjacency matrix are real numbers (Doosti, 2011).

Def. 12: Suppose that Γ is a graph, the formed set from the determined values of adjacency matrix A with its repetition is called the Γ spectrum. If $\lambda_0>\lambda_1>\cdots>\lambda_{s-1}$ be the entire different determined values of A and $m(\lambda_0)=m_0$, $m(\lambda_1)=m_1,\cdots,m(\lambda_{s-1})=m_{s-1}$ be its repetition, in this case we indicate the Γ spectrum as following (Nazary, 2016).

$$Spec_A(\Gamma) = \begin{pmatrix} \lambda_0 & \lambda_1 & \cdots & \lambda_{s-1} \\ m_0 & m_1 & \cdots & m_{s-1} \end{pmatrix}$$

Def. 13: Two graphs are symmetric, if they include equal spectra. In other words, they consist equal determined polynomials (Malay, 2002).

Example 6: Obtain the determined values and spectrum of the complete graph K_4.

Solution: Suppose A is adjacency matrix of K_4, in this case it is written as:

$$A = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \Rightarrow \chi(K_4,\lambda) = \det(\lambda I_4 - A) = \begin{pmatrix} \lambda & -1 & -1 & -1 \\ -1 & \lambda & -1 & -1 \\ -1 & -1 & \lambda & -1 \\ -1 & -1 & -1 & \lambda \end{pmatrix}$$

We write the row elements sum of the above each matrix row in its first row position:

$$\chi(K_4,\lambda) = \begin{pmatrix} \lambda-3 & \lambda-3 & \lambda-3 & \lambda-3 \\ -1 & \lambda & -1 & -1 \\ -1 & -1 & \lambda & -1 \\ -1 & -1 & -1 & \lambda \end{pmatrix} = (\lambda-3)\begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & \lambda & -1 & -1 \\ -1 & -1 & \lambda & -1 \\ -1 & -1 & -1 & \lambda \end{pmatrix}$$

$$\chi(K_4,\lambda) = (\lambda-3)\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & \lambda+1 & 0 & 0 \\ 0 & 0 & \lambda+1 & 0 \\ 0 & 0 & 0 & \lambda+1 \end{pmatrix} = (\lambda-3)(\lambda+1)^3 = 0$$

Therefore, $\lambda_0 = 3$ with repetition of 1 and $\lambda_1 = -1$ with repetition of 3 are determined values of K_4 and its spectrum is:

$$Spec_A(K_4) = \begin{pmatrix} 3 & -1 \\ 1 & 3 \end{pmatrix}$$

Example 7: Obtain the determined values and spectrum of the complete graph K_n

Solution: Suppose A is the adjacency matrix of K_n, in this case it is written:

$$A = \begin{pmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 1 & \cdots & 1 \\ 1 & 1 & \ddots & \cdots & 1 \\ \vdots & \vdots & \vdots & 0 & \vdots \\ 1 & 1 & 1 & \cdots & 0 \end{pmatrix}$$

$$\chi(K_n,\lambda) = \det(\lambda I_n - A) = \begin{pmatrix} \lambda & -1 & -1 & \cdots & -1 \\ -1 & \lambda & -1 & \cdots & -1 \\ -1 & -1 & \lambda & \cdots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \cdots & \lambda \end{pmatrix}$$

We write the sum of row elements of the above each matrix row in its first row position:

$$\chi(K_n,\lambda)=\begin{pmatrix} \lambda+1-n & \lambda+1-n & \lambda+1-n & \cdots & \lambda+1-n \\ -1 & \lambda & -1 & \vdots & -1 \\ -1 & -1 & \lambda & \vdots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \cdots & \lambda \end{pmatrix}$$

$$\chi(K_n,\lambda)=(\lambda+1-n)\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ -1 & \lambda & -1 & \cdots & -1 \\ -1 & -1 & \lambda & \cdots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \cdots & \lambda \end{pmatrix}$$

$$\chi(K_n,\lambda)=(\lambda+1-n)\begin{pmatrix} 1 & 1 & 1 & \vdots & 1 \\ 0 & \lambda+1 & 0 & \vdots & 0 \\ 0 & 0 & \lambda+1 & \vdots & 0 \\ \cdots & \cdots & \cdots & \ddots & \vdots \\ 0 & 0 & 0 & \vdots & \lambda+1 \end{pmatrix}$$

$$\chi(K_n,\lambda)=(\lambda+1-n)(\lambda+1)^{n-1}=0$$

Therefore, $\lambda_0=n-1$ with repetition of 1 and $\lambda_1=-1$ with repetition of $(n-1)$ are determined values of K_4 and its spectrum is:

$$Spec_A(K_n)=\begin{pmatrix} n-1 & -1 \\ 1 & n-1 \end{pmatrix}$$

Lima 1: Suppose that $\chi(\Gamma,\lambda)=\lambda^n+c_1\lambda^{n-1}+c_2\lambda^{n-2}+c_3\lambda^{n-3}+\cdots+c_n$ is determined polynomial adjacency matrix of Γ graph, in this case:

1. $c_1=0$
2. $-c_2$ is equal to the number Γ edges
3. $-c_3$ is equal to the double number of the Γ triangles.

Theorem 1: Suppose that Γ is n vertex, m edge and $\lambda_1\geq\lambda_2\geq\cdots\geq\lambda_n$ is its determined values, in this case:

1. $\sum_{i=1}^{n}\lambda_i=0$
2. $\sum_{i=1}^{n}\lambda_i^2=2m$
3. $\sum_{i=1}^{n}\lambda_i^3=6t$, in which t indicates number Γ triangles (Watandoost, 2012).

Prove

1. Suppose A is the adjacency matrix of the Γ graph, hence there is invertible matrix of P, since $D = P^{-1}AP$, therefore it can be written:

$$tr(D) = tr\left(P^{-1}AP\right) = tr(A) = \sum_{i=1}^{n} \lambda_i = 0 \Rightarrow \sum_{i=1}^{n} \lambda_i = 0$$

2. Since $D^2 = P^{-1}A^2P$, therefore it can be written:

$$tr\left(D^2\right) = tr\left(P^{-1}A^2P\right) = tr\left(A^2\right) = \sum_{i=1}^{n} \lambda_i^{\,2}$$

The above equality is equal to the number of closed paths with 2 length in Γ. Since closed paths in length only occur in edges and each edge determines two closed paths. Therefore, we have:

$$\sum_{i=1}^{n} \lambda_i^{\,2} = 2m$$

3. Since $D^3 = P^{-1}A^3P$. Therefore, it can be written:

$$tr\left(D^3\right) = tr\left(P^{-1}A^3P\right) = tr\left(A^3\right) = \sum_{i=1}^{n} \lambda_i^{\,3}$$

The above equality is equal to the number of closed paths with 3 length in Γ. Since closed paths in length of 3 only occur in triangle and each triangle determines six closed paths. Therefore, we have:

$$\sum_{i=1}^{n} \lambda_i^{\,3} = 6t$$

Def. 14: Suppose $S = \left(s_{ij}\right)$ is a square matrix, as $s_{ij} = s_{1,j-i+1}$ and $j-i+1$ equivalent to n, i and j substitute, hence, S is called cycle matrix. By considering the above definition we can determine a cycle matrix by the first row clarity (Tongue, 2005).

Example 8: We write a cycle matrix 4×4 with first row $(0,1,1,1)$ (Nazary, 2015).

$$S = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

Def. 15: The Γ graph is called cycle, if its adjacency matrix be cyclic (Anderson & Shapiro, 2003).

Def. 16: Suppose that Γ is a graph with set vertices of $V = \{v_1,\cdots,v_n\}$ and $E = \{e_1,\cdots,e_m\}$ is its edges. The Γ edge graph which denotes with $L(\Gamma)$ notation is a graph with set vertices of $V = \{v_1,\cdots,v_n\}$, as the two vertices v_i and v_j are called adjacent in $L(\Gamma)$, if e_i and e_j be as two edges in Γ adjacent (Nazary, 2015).

Example 9: The complete adjacency graphmatrices, obtain the $L(K_3)$ and edge graph K_3 occurrence matrix of K_3 and $Diag\left(K_3\right)$.

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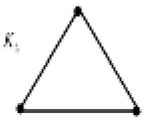
Solution: Suppose A and A_L are the adjacency matrices of K_3 and $L(K_3)$, X is the occurrence matrix of K_3, in this case, we consider the complete graph K_3, we will find the matrices.

K_3



$$A = A_L = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \ X = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \text{and } Diag(K_3) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

Lima 2: Suppose Γ is a graph with n vertex and edge m and X is its occurrence matrix. On other hand, suppose A and A_L are adjacency matrices of Γ and $L(\Gamma)$, in this case:

1. $X^tX = A_L + 2I_m$

2. $XX^t = A + Diag(\Gamma)$ whereas $Diag(\Gamma) = \begin{pmatrix} \deg(v_1) & 0 & \cdots & 0 \\ 0 & \deg(v_2) & \cdots & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \cdots & 0 & \deg(v_n) \end{pmatrix}$

3. If Γ, -r be regular then $XX^t = A + rI_n$.

Prove

1. $(X^tX)_{ij} = X^t\ i^{\text{th}}\text{row} \cdot X\ j^{\text{th}}\text{column} = X\ i^{\text{th}}\text{row} \cdot X\ j^{\text{th}}\text{column}$

$$(X^tX)_{ij} = \begin{cases} 1 & if\ e_i\ and\ \mathrm{e}_j\ include\ joint\ vertices \\ 0 & Otherwise \end{cases}$$

$$(X^tX)_{ii} = X^t\ i^{\text{th}}\text{column} \cdot X\ i^{\text{th}}\text{column} = X\ i^{\text{th}}\text{column} \cdot X\ i^{\text{th}}\text{column}$$

$$\cdot X = 2$$

$$(A_L + 2I_m)_{ij} = (A)_{ij} + 0 = \begin{cases} 1 & if\ e_i\ and\ \mathrm{e}_j\ include\ joint\ vertices \\ 0 & Otherwise \end{cases}$$

$$(A_L + 2I_m)_{ii} = 0 + 2$$

$$\Rightarrow X^tX = A_L + 2I_m$$

2. $(XX^t)_{ij} = X\ i^{\text{th}}\text{row.} \cdot X^t\ j^{\text{th}}\text{column} = X\ i^{\text{th}}\text{row} \cdot X\ j^{\text{th}} \cdot X$

$$(XX^t)_{ij} = \begin{cases} 1 & if\ v_i\ and\ \mathrm{v}_j\ be\ adjacent \\ 0 & Otherwise \end{cases}$$

$$(XX^t)_{ii} = i^{\text{th}}\text{row} \cdot X\ i^{\text{th}}\text{column} = = \deg(v_i)$$

$$\left(A+Diag(\Gamma)\right)_{ij}=\begin{cases}1 & if\ v_i\ and\ v_j\ be\ adjacent\\ 0 & Otherwise\end{cases}$$

$$\left(A+Diag(\Gamma)\right)_{ii}=0+\deg(v_i)\Rightarrow XX^t=A+Diag(\Gamma)$$

3. From 1 and 2 it can be easily resulted.

Def. 17: The matrix XX^t is called Laplacian matrix without Γ notation (Biggs, 1993).

Result 1: If the graphs Γ_1 and Γ_2 in terms of Laplacian matrix are symmetric, then the graphs $L(\Gamma_1)$ edge and $L(\Gamma_2)$ also in terms of adjacency matrix are symmetric.

Result 2: Suppose that Γ is a - r regular graph with n vertex and m edge and $Spec_A(\Gamma)=\begin{pmatrix}r & \lambda_1 & \cdots & \lambda_{s-1}\\ 1 & m_1 & \cdots & m_{s-1}\end{pmatrix}$, in this case we have:

$$Spec_{A_L}\left(L(\Gamma)\right)=\begin{pmatrix}2r-2 & r-2+\lambda_1 & \cdots & r-2+\lambda_{s-1} & -2\\ 1 & m_1 & \cdots & m_{s-1} & m-n\end{pmatrix}$$

Example 10: Obtain the complete graph K_3 spectrum and $L(K_3)$ edge graph.

Solution: Suppose the A and A_L are the adjacency matrices of K_3 and $L(K_3)$, in this case it can be written:

$$A=A_L=\begin{pmatrix}0&1&1\\1&0&1\\1&1&0\end{pmatrix}\Rightarrow \chi(K_3,\lambda)=\det(\lambda I_3-A)=\begin{vmatrix}\lambda&-1&-1\\-1&\lambda&-1\\-1&-1&\lambda\end{vmatrix}$$

We write the sum of row elements of the above each matrix row in its first position:

$$\chi(K_3,\lambda)=\begin{vmatrix}\lambda-2&\lambda-2&\lambda-2\\-1&\lambda&-1\\-1&-1&\lambda\end{vmatrix}=(\lambda-2)\begin{vmatrix}1&1&1\\-1&\lambda&-1\\-1&-1&\lambda\end{vmatrix}$$

$$\chi(K_3,\lambda)=(\lambda-2)\begin{vmatrix}1&1&1\\0&\lambda+1&0\\0&0&\lambda+1\end{vmatrix}=(\lambda-2)(\lambda+1)^2=0$$

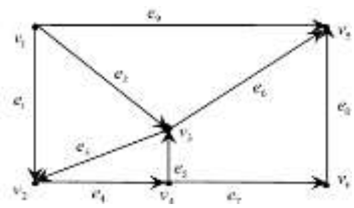
Therefore, $\lambda_0=2$ with repetition of 1 and $\lambda_1=-1$ with repetition of 2 are the determined values of K_3 and its spectrum is:

$$Spec_A(K_3)=Spec_{A_L}\left(L(K_3)\right)=\begin{pmatrix}2&-1\\1&2\end{pmatrix}$$

Def. 18: Suppose Γ is a directed graph with $V=\{v_1,\cdots,v_n\}$ set vertices and $E=\{e_1,\cdots,e_m\}$ is its set edges. The matrix $D=(d_{ij})$ is called the directed graph occurrence matrix of Γ , which is a $n\times m$ rectangular matrix and its d_{ij} th element is:

$$d_{ij} = \begin{cases} 1 & if\ \ the\, e_j\ direction\ is\ toward\ v_i\, vertex \\ -1 & if\ \ the\, e_j\ direction\ is\ not\ toward\ v_i \\ 0 & Otherwise \end{cases}$$

Example 11: Obtain the occurrence matrix of the following directed graph (Murty, 2006).



Solution: The above directed graph occurrence matrix is:

$$D = \begin{pmatrix} -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \end{pmatrix}$$

The Directed Graph Occurrence Matrix Properties: Suppose Γ is directed graph with n vertex and m edge and X is its occurrence matrix, the basic properties of X are:

1. The X elements only consist of $-1, 0, 1$.
2. The sum of each X column is equal to zero.
3. The positive sum of elements in each X row is equal to the input degree of that vertex and the eigenvalue of negative elements sum in each X row is equal to the output degree of its vertex.
4. If entire elements of a X row is non-negative, its related vertex is edge ending row, and if the entire elements of a X row is non-positive, its related row edge is starting row (Maqsoumi & Nekokar, 2014).

Lima 3: Suppose Γ is a graph, it is resulted from adjacency matrices spectrum and Laplacian without notation as:

1. Number of Γ vertices
2. Number of Γ edges
3. The number of closed paths with alternative length in Γ only result from adjacency matrix (Makes & Cvetkovic, 1976).

Def. 19: The Γ graph is also called whole graph, if entire determined values of its adjacency matrix are integers.

Example 12: The K_n complete graph is a whole graph (Sharp, 1990).

Lima 4: The C_n cycle is a whole graph. If and only if $n \in \{3, 4, 6\}$.

Lima 5: The P_n path is a whole graph. If and only if $n \in \{2\}$.

Theorem 2: There is exactly 263 whole graph with maximum 11 vertices.

Conclusion

We found out the following result from "Role of rectangular and square matrices in graph theory":

1. In Γ adjacency matrix graph, each element of it, is zero or one and its main diagonal elements are zeroes.
2. The Γ graph adjacency matrix is a $n \times n$ square matrix.
3. The Γ graph adjacency matrix is a corresponding matrix.
4. The adjacency matrix in directed and undirected graphs are similar.
5. The cycle matrix is a square matrix.
6. The occurrence matrix of the Γ directed and undirected graph is a $n \times m$ rectangular matrix.
7. In occurrence matrix of theundirected graph zeroes and one consist its elements, as the sum of each Γ row elements is the vertex degree of that row and the sum of each column elements is equal to 2.
8. In Γ directed graph occurrence matrix, its elements only consist of $-1, 0, 1$, as each column elements sum are equal to zero, the positive elements sum in each row are equal to input degree of that vertex and the eigenvalue of negative elements sum in each row is equal to the output degree of its vertex. If entire elements of a row is non-negative, its related vertex is edge ending row, and if the entire elements of a row is non-positive, its related row edge is starting row.

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